MURA 68

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Investigation of the Effect of Position of Straight Sections with Application to the Mark V Spirally Ridged Accelerator

by

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Errata at back

The purpose of this paper is to estimate the effect of straight sections on σ_r and σ_z in a spirally ridged accelerator. For orbits of differing radii, the straight sections occur at different positions relative to the positive and negative gradient regions along the path of the particle. The problem is treated by testing the effect of the straight section position in a magnet obeying a Hill's equation with the same σ_r and σ_z of the spirally ridged accelerator.

For Hill's equation, we consider the following cases.

We see that the transformation matrix across a complete sector is $M_o = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and $Tr\ (M_o)$ is $ae + bg + cf + dh$.

Case II: Straight section between a focussing and a defocussing sector. One straight section per complete sector. The transformation matrix for the straight section is $\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}$, where θ is the angle in radians subtended by the straight section. Let M_1 be the matrix of the transformation in the case where the straight section follows the defocussing sector and M_2 be the matrix for the transformation in the case where the straight section follows the focussing sector.

$$M_1 = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$Tr(M_1) = Tr\ M_o + \theta\ (ga + hc)$

$Tr(M_2) = Tr\ M_o + \theta (ce + dg)$

So $Tr(M_2) = Tr(M_1)$, since $a = d$ and $e = h$.

Case III: Straight section in a defocussing sector.

$$M_3 = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 & f_2 \\ g_2 & h_2 \end{pmatrix}$$

where $\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e_2 & f_2 \\ g_2 & h_2 \end{pmatrix} \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}$

$$M_3 = \theta\begin{pmatrix} ag_1e_2 + bg_1g_2 + ch_1e_2 + dh_1g_2 & g_1af_2 + bg_1h_2 + ch_1f_2 + dh_1h_2 \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e_2 & f_2 \\ g_2 & h_2 \end{pmatrix} = \theta\Delta M_3 + \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e_2 & f_2 \\ g_2 & h_2 \end{pmatrix}$$

$$Tr(M_3) = \theta(ag_1e_2 + bg_1g_2 + ch_1e_2 + dh_1g_2) + Tr(M_o) = \theta\, Tr(\Delta M_3) + Tr(M_o)$$

Case IV: Straight section in the center of a focussing sector.

$$M_4 = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$

$$M_4 = \theta\begin{pmatrix} ec_1a_2 + fc_1c_2 + gd_1a_2 + hd_1c_2 & ec_1b_2 + fc_1d_2 + gd_1b_2 + hd_1d_2 \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \theta\Delta M_4 + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$TrM_4 = \theta\,(ec_1a_2 + fc_1c_2 + gd_1a_2 + hd_1c_2) + Tr(M_o) = \theta\, Tr(\Delta M_4) + Tr(M_o)$$

At this point, we might note that M_3 and M_4 are sufficiently general that M_1 and M_2 could be derived from either M_3 or M_4. For example in M_4, if we let the straight section move toward the end of the focussing sector, $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, M_4 becomes $\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$. This is merely M_2. However, the simple nature of M_1 or M_2 makes it somewhat impractical to ignore M_1 and deal with the more complex M_3 or M_4.

We shall now investigate the effect of adding more straight sections. In Case V, we shall consider the case where there are two straight sections per complete sector. One will be encountered when a particle passes from a focussing to a defocussing sector, and the othere will be encountered when a particle passes from a defocussing sector to a focussing sector.

If we had started at a different point to write M_1, we would have

$$M_1' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \theta\Delta M_1' + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Since $Tr(M_1) = Tr(M_1')$, $Tr(\Delta M_1') = Tr(\Delta M_1)$

$$M_5 = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}(M_1') = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}\theta\Delta M_1' + M_2$$

which

$$= \begin{pmatrix} \theta & \theta^2 \\ 0 & \theta \end{pmatrix}\Delta M_1' + M_2$$

If $\theta^2 \cong 0$,

$$M_5 \cong \theta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Delta M_1' + M_2$$

$$\text{So } Tr(M_5) = \theta\, Tr(\Delta M_1') + Tr(M_2)$$

$$= \theta(2Tr(\Delta M_1)) + Tr(M_o)$$

Case VI: Straight sections in focussing and defocussing sectors. Similarly to the development in case V, assume that we had started at a different point to write M_3.

$$M_3' = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} e_2 & f_2 \\ g_2 & h_2 \end{pmatrix}\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$= \theta\Delta M_3' + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Again, since $Tr(M_3) = Tr(M_3')$, $Tr(\Delta M_3) = Tr(\Delta M_3')$

$$M_6 = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} (M_3')$$

$$= \begin{pmatrix} \theta & \theta^2 \\ 0 & \theta \end{pmatrix} \Delta M_3' + M_4 .$$

Again making the approximation that $\theta^2 \cong 0$, we have

$$Tr(M_6) \cong Tr(\theta \Delta M_3') + Tr(\theta\Delta M_4) + Tr(M_o)$$

$$\cong Tr(\theta \Delta M_3) + Tr(\theta\Delta M_4) + Tr(M_o)$$

For Mark V we have Laslett's linearized equations:

$$\frac{d^2 V}{d\theta^2} + \left(k + 1 - \frac{f^2}{2\lambda^2N^2} + \frac{f}{\lambda} \cos(N\theta) + \frac{f^2}{2\lambda^2N^2}\cos(2N\theta)\right)V = 0$$

$$\frac{d^2 Z}{d\theta^2} + \left(-k + \frac{f^2}{2\lambda^2N^2} - \frac{f}{\lambda} \cos(N\theta) - \frac{f^2}{2\lambda^2N^2}\cos(2N\theta)\right)Z = 0$$

where the V motion is radial and the Z motion is vertical. So for these parameters:

$k = 150$

$\nu_r = 12.3$

$\nu_z = 6.2$

$N = 37$

$f/\kappa = 504$

$f = .25$

$$\frac{d^2 V}{d\,\theta^2} + \left\{57 + 504\cos N\theta + 93\cos 2N\theta\right\} V = 0$$

$$\frac{d^2 Z}{d\,\theta^2} + \left\{-57 - 504\cos N\theta - 93\cos 2N\theta\right\} Z = 0$$

We want to substitute Hill's equations for these equations with about the same σ_r and σ_z so that the previous results on straight sections can be applied to these imitation Mark V equations. We ignore the second harmonic because the smooth approximation says $\bar{f} + \sum_{j=1}^{\infty} \frac{A_j^2}{2j^2N^2}$ is the force constant. This means that the contribution of the second harmonic is less than 1%. This would give us approximately

$$\frac{d^2 V}{d\,\theta^2} + \begin{Bmatrix} 416 \\ -300 \end{Bmatrix} V = 0$$

$$\frac{d^2 Z}{d\,\theta^2} + \begin{Bmatrix} -416 \\ +300 \end{Bmatrix} Z = 0.$$

Using the Wayne Tables, we see from this that we should change to the following parameters so that $\sigma_r \sim 2\pi/3$ and $\sigma_z \sim \pi/3$

$$\frac{d^2 V}{d\,\theta^2} + \begin{Bmatrix} +386 \\ -300 \end{Bmatrix} V = 0$$

$$\frac{d^2 z}{d \theta^2} + \begin{Bmatrix} -386 \\ +300 \end{Bmatrix} z = 0$$

This gives

$$\psi_1 = \sqrt{300}\, \theta_1 = \sqrt{300}\, \frac{\pi}{37} = 84.2^\circ$$

$$\psi_2 = \sqrt{386}\, \theta_2 = \sqrt{386}\, \frac{\pi}{37} = 95.7^\circ$$

$$\sigma_r = 119^\circ$$

$$\sigma_z = 56.6^\circ$$

Using these values, with one straight section per complete sector

$$\mathrm{Tr}(\theta \Delta M_1) = -\theta (48.34)$$

If the straight section is 20 cm. long, and the machine radius is 10^4 cm., then $\theta = 20 \times 10^{-4}$,

$$\sigma_r = 122.21^\circ$$

$$\nu_r = 12.59$$

$$\mathrm{Tr}(\theta \Delta M_3) = -\theta (25.0)$$

$$\sigma_r = 120.65^\circ$$

$$\nu_r = 12.40$$

$$\mathrm{Tr}(\theta \Delta M_4) = -\theta (3.5)$$

$$\sigma_r = 119.23^\circ$$

$$\nu_r = 12.28$$

For Z oscillations,

$$\mathrm{Tr}(\theta\,\Delta\,M_1) = -\theta(42.3)$$

$$\sigma_z = 59.95^\circ$$

$$\nu_z = 6.16$$

$$\mathrm{Tr}(\theta\,\Delta\,M_3) = -7.74\,\theta$$

$$\sigma_z = 57.13^\circ$$

$$\nu_z = 5.87$$

$$\mathrm{Tr}(\theta\,\Delta\,M_4) = -7.1\,\theta$$

$$\sigma_z = 57.08^\circ$$

$$\nu_z = 5.86$$

Now, in the case of two straight sections per complete sector, for radial oscillations

$$\mathrm{Tr}\,(\theta\,\Delta\,M_5) = -(48.34\ )2\theta$$

$$\sigma_r = 125.56^\circ$$

$$\nu_r = 12.9$$

$$\mathrm{Tr}(\theta\,\Delta\,M_6) = -(3.5 + 25.0[illegible])\,\theta$$

$$\sigma_r = 120.88^\circ$$

$$\nu_r = 12.41$$

For vertical oscillations

$$\mathrm{Tr}(\theta\Delta\,M_5) = -(42.3)2\theta$$

$$\sigma_z = 62.23^\circ$$

$$\nu_z = 6.4$$

$$\mathrm{Tr}\,(\theta \Delta M_6) = -(7.74 + 7.1)\,\theta$$

$$\sigma_z = 57.61^\circ$$

$$\nu_z = 5.93$$

We see that, unfortunately, we cross half-integral resonances for the radial oscillations and integral resonances for vertical oscillations. It would be nice to find a way of changing machine parameters for certain radii such that σ_r and σ_z would not vary over such a wide range. However, if we control σ_r by lowering k, we make the variation of σ_z worse. It might be possible, however, to remedy this by also making a small change in the AG term.

Errata 68

$$\frac{d^2 Z}{d\, \theta^2} + \begin{Bmatrix} -386 \\ +300 \end{Bmatrix} Z = 0$$

This gives

$$\psi_1 = \sqrt{300}\, \theta_1 = \sqrt{300}\, \frac{\pi}{37} = 84.2^\circ$$

$$\psi_2 = \sqrt{386}\, \theta_2 = \sqrt{386}\, \frac{\pi}{37} = 95.7^\circ$$

$$\sigma_r = 119^\circ$$

$$\sigma_z = 56.6^\circ$$

Using these values, with one straight section per complete sector

$$Tr(\theta\, \Delta\, M_1) = - \theta\, (48.34)$$

If the straight section is 20 cm. long, and the machine radius is 10^4 cm., then $\theta = 20 \times 10^{-4}$,

$$\sigma_r = 122.21^\circ$$

$$\nu_r = 12.59$$

$$Tr(\theta \Delta M_3) = - \theta (25.0)$$

$$\sigma_r = 120.65^\circ$$

$$\nu_r = 12.40$$

$$Tr(\theta\, \Delta\, M_4) = - \theta (3.5)$$

$$\sigma_r = 119.23^\circ$$

$$\nu_r = 12.28$$

For Z oscillations,

$$\mathrm{Tr}(\theta \Delta M_1) = -\theta(42.3)$$

$$\sigma_z = 59.95^{\circ}$$

$$\nu_z = 6.16$$

$$\mathrm{Tr}(\theta \Delta M_3) = -7.74\theta$$

$$\sigma_z = 57.13^{\circ}$$

$$\nu_z = 5.87$$

$$\mathrm{Tr}(\theta \Delta M_4) = -7.1\theta$$

$$\sigma_z = 57.08^{\circ}$$

$$\nu_z = 5.86$$

Now, in the case of two straight sections per complete sector, for radial oscillations

$$\mathrm{Tr}(\theta \Delta M_5) = -(48.34)2\theta$$

$$\sigma_r = 125.56^{\circ}$$

$$\nu_r = 12.9$$

$$\mathrm{Tr}(\theta \Delta M_6) = -(3.5 + 25.0)\theta$$

$$\sigma_r = 120.88^{\circ}$$

$$\nu_r = 12.41$$

For vertical oscillations

$$\mathrm{Tr}(\theta \Delta M_5) = -(42.3)2\theta$$

$$\sigma_z = 62.23^{\circ}$$

$$\nu_z = 6.4$$